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ELECTRON-OPTICAL PROPERTIES OF HOMOGENEOUS MAGNETIC AND RADIAL ELECTRIC FIELDS

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1. Introduction

For various purposes it is often desired to change the direction or alter the characteristics of a given beam of ions. This can be done with various combinations of magnetic and electric fields, of which a homogeneous magnetic field and a radial electric field, such as is found inside an electrostatic deflector, are among the easiest to produce. It is the purpose of this report to summarize the optical properties of these two types of fields, and of various combinations thereof. It is always assumed that the electric and magnetic fields are perpendicular wherever they are superimposed, are plane bounded, and that the beam is in what shall be called the horizontal plane, which is defined as the plane perpendicular to the magnetic field and/or parallel to the electric field. The discussion will be confined to effects of the first order in the deviations in position, direction, velocity, and mass of the ions from reference values.

The theory of focussing effects in the horizontal plane is based on a paper by R. Hertzog,[1] who was interested chiefly in applications to mass spectroscopy. Parts of his discussion have been generalized somewhat; further the focussing effects in the vertical plane (the plane parallel to the magnetic field and/or perpendicular to the electric field) are given. All optical properties are expressed by equations which show the analogy with thick lenses, and so far as possible Hertzog's notation has been used. Lastly some discussion of the application of the theory to the problem of injection into the bevatron has been included.

[1] R. Hertzog, Zeits f. Phys. 89, 447 (1934)

2. The Orbital Equations for Fields Bounded by Planes Normal to the Incoming and Outgoing Rays

Consider an electromagnetic field confined to a wedge shaped region, shown in Fig. 1 as region III.

Regions I and II are field free. In region III, bounded by the lines Py' and Py", is a uniform magnetic field H perpendicular to the plane of the paper, directed upwards or downwards according as the ion is positive or negative respectively, and an electric field produced by two cylindrical condenser plates concentric with P of radius R_1 and R_2 (where $R_1 > R_2$) with a potential difference X. If the potentials of the cylinders are so adjusted that the circle of radius a has zero potential, the potential and field at a radius r are given by[2]

$$V(r) = \frac{X}{\ln \frac{R_1}{R_2}} \ln \frac{r}{a} \approx \frac{X}{a \ln \frac{R_1}{R_2}} (r - a)$$

$$\vec{E}(r) = -\frac{1}{r} \frac{X}{\ln \frac{R_1}{R_2}} \frac{\vec{r}}{r}. \tag{1}$$

H and X are of such magnitude and sense that an ion with charge e, velocity v_o, and rest mass M_o incident normally to Py' at O' will follow a circular path of radius a about P, emerging at O", in a direction normal to Py", that is along x".

For this particle, the radius of curvature a is given by

$$\frac{M_o v_o^2}{a\sqrt{1 - v_o^2/c^2}} = \frac{eX}{a \ln \frac{R_1}{R_2}} + e\frac{H}{c} v_o \tag{2}$$

or

$$\frac{1}{a} = \frac{1}{a_e} + \frac{1}{a_m} \tag{2'}$$

where

$$a_e = \frac{M_o v_o^2}{\sqrt{1 - v_o^2/c^2}} \frac{a \ln \frac{R_1}{R_2}}{eX}, \qquad a_m = \frac{M_o v_o}{\sqrt{1 - v_o^2/c^2}} \frac{c}{eH} \tag{3}$$

2 Smythe § 2.04

are the radii of curvature that would exist if the electric and magnetic fields were present alone, respectively.* If the curvatures are opposed the smallest of a_e or a_m is taken positive, the largest negative.

Consider now an ion with velocity and rest mass

$$v = v_o\ (1 + \lambda)$$
$$M = M_o\ (1 + \gamma) \tag{4}$$

whose path in regions I and II, with respect to axes x'y', and x"y" is

$$y' = y_1 + \alpha' x' \tag{5}$$

$$y'' = y_2 + \alpha'' x''. \tag{6}$$

It is shown in Appendix I that the path of this ion in region III, to the first order in λ, γ, α', $\frac{y'}{a}$, and $z = \frac{r - a}{a}$, is given by

$$r - a = a\left[-\frac{\alpha'}{K}\sin K\varphi + \delta(1 - \cos K\varphi) + \frac{y_1}{a}\cos K\varphi\right] \tag{7}$$

where K and δ are defined by

$$K^2 = 1 + \left(\frac{a}{a_e}\right)^2 (1 - \beta_o^2) \tag{8}$$

$$K^2\delta = \gamma + \lambda\left(1 + \frac{a}{a_e} + \frac{\beta_o^2}{1 - \beta_o^2}\right) \equiv \gamma + B\lambda \tag{9}$$

and $\beta_o = \frac{v_o}{c}$.

Hence

$$y_2 = r(\Phi) - a = a\left[-\frac{\alpha'}{K}\sin K\Phi + \delta\ (1 - \cos K\Phi) + \frac{y_1}{a}\cos K\Phi\right] \tag{10}$$

$$\alpha'' = \frac{1}{a}\left(\frac{dr}{d\varphi}\right)_\Phi = -\alpha' \cos K\Phi + \delta K \sin K\Phi - \frac{y_1}{a} K \sin K\Phi \tag{11}$$

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* If $V(a) \neq 0$ the analysis is unchanged provided that in equations (2) and (3) v_o is replaced by $\bar{v}_o = v_o\left[1 - (1 - \beta_o^2)^{3/2}\frac{e\,V(a)}{M_o\,v_o^2}\right]$, the velocity of the same particle at potential $V(a)$.

so that from (6)

$$y'' = y_2 + \alpha''x'' = a\left[-\frac{\alpha'}{K}\sin K\Phi + \delta(1 - \cos K\Phi) + \frac{y_1}{a}\cos K\Phi\right]$$
$$+ x''\left[-\alpha' \cos K\Phi + \delta K \sin K\Phi - \frac{y_1}{a} K \sin K\Phi\right]. \qquad (12)$$

3. Generalization to Arbitrary Plane Boundaries

Suppose now that the boundaries of region III* are not the y' and y'' axes, but are inclined at angles ε' and ε'' to these axes, respectively, as shown in Fig. 2, where $x'O'O''x''$ again represents the reference orbit.

n' and n'' are the normals to the planes bounding the field. All quantities are to be taken as positive when the arrangement is as shown in Fig. 1 or Fig. 2. Ω is the angle between the field-bounding planes and is to be taken as positive when the intersection of the planes is on the same side of the orbit as P, the center of the reference ray's circle $O'O''$. It is related to Φ, ε', and ε'' by

$$\Omega = \Phi - \varepsilon' - \varepsilon''. \qquad (13)$$

Further, if c' is the distance between O' and the intersection of the field bounding planes, then it may be shown geometrically that

$$\sin \varepsilon'' = \frac{c'}{a}\sin\Omega - \sin(\Omega + \varepsilon'). \qquad (13')$$

An ion described by (4) which passes through the point $x' = \ell'$, $y' = b'$ with angle α' will have a path in I given by

$$y' = b' + (x' - \ell')\alpha'. \qquad (14)$$

It will enter the field at the point Q', leave it at Q'', and will behave as if it had entered a field bounded by PQ' and PQ''. Hence its orbit is given with respect to Cartesian axes $\bar{x}'\bar{y}'$ in I and $\bar{x}''\bar{y}''$ in II (see Fig. 2) by equations (5),(6) and (12), with bars placed over all variables.

The relations between barred and unbarred variables, to first order, are†

---

* The effect of the fringing fields is discussed in Appendix C.

† For brevity the notation tn is used for tangent, instead of tan.

$$
\begin{aligned}
\bar{x} &= x + a\,\Delta\Phi & \bar{\Phi} &= \Phi - \Delta\Phi' - \Delta\Phi'' \\
\bar{y} &= y & \Delta\Phi' &= \frac{y_1}{a}\tan\varepsilon' \\
\bar{\alpha} &= \alpha - \Delta\Phi & \Delta\Phi'' &= \frac{y_2}{a}\tan\varepsilon''.
\end{aligned}
\tag{15}
$$

From (14) it is seen that

$$y_1 = b' - \alpha'\ell'. \tag{16}$$

Substituting equations (15) and (16) in (12), where all variables in the latter are barred, and discarding terms of higher than first order, there is obtained for the path of the outgoing ion the equation

$$
\begin{aligned}
y'' = b' &\left\{ -x'' \frac{\cos K\Phi}{a}\left(K\tan K\Phi - \tan\varepsilon' - \tan\varepsilon'' - \tan\varepsilon'\tan\varepsilon''\frac{\tan K\Phi}{K}\right)\right. \\
&\qquad\left. + \cos K\Phi\left(1 + \tan\varepsilon'\frac{\tan K\Phi}{K}\right)\right\} \\
+ \alpha' &\left\{ x''\left[\ell'\frac{\cos K\Phi}{a}\left(K\tan K\Phi - \tan\varepsilon' - \tan\varepsilon'' - \tan\varepsilon'\tan\varepsilon''\frac{\tan K\Phi}{K}\right)\right.\right. \\
&\qquad\left. - \cos K\Phi\left(1 + \tan\varepsilon''\frac{\tan K\Phi}{K}\right)\right] \\
&\qquad\left. - \cos K\Phi\left[\ell'\left(1 + \tan\varepsilon'\frac{\tan K\Phi}{K}\right) + a\frac{\tan K\Phi}{K}\right]\right\} \\
+ \delta &\left\{ x''\left[K\sin K\Phi + (1 - \cos K\Phi)\tan\varepsilon''\right] + a\,(1 - \cos K\Phi)\right\}.
\end{aligned}
\tag{17}
$$

This is an important equation, since from it all the optical properties of the system may be deduced.

4. Optical Properties of the System

The point (ℓ', b') is called the object point, and ℓ' the object distance. All ions from (ℓ', b') with the same δ will converge at a point (ℓ'', b''), called the image point. The image distance ℓ'' is that x'' for which the dependence of y'' on α' vanishes. From (17) it is seen that ℓ'' is given by

$$\ell'' = a\,\frac{\ell'\left(1 + \tan\varepsilon'\frac{\tan K\Phi}{K}\right) + a\frac{\tan K\Phi}{K}}{\ell'\left(K\tan K\Phi - \tan\varepsilon' - \tan\varepsilon'' - \tan\varepsilon'\tan\varepsilon''\frac{\tan K\Phi}{K}\right) - a\left(1 + \tan\varepsilon''\frac{\tan K\Phi}{K}\right)}. \tag{18}$$

The analogy with a cylindrical lens is facilitated by the introduction of the following variables

$$f \equiv a\left[\cos K\Phi\left(K\ \mathrm{tn}\ K\Phi - \mathrm{tn}\ \varepsilon' - \mathrm{tn}\ \varepsilon'' - \mathrm{tn}\ \varepsilon'\ \mathrm{tn}\ \varepsilon''\ \frac{\mathrm{tn}\ K\Phi}{K}\right)\right]^{-1} \tag{19}$$

$$g' \equiv f\cos K\Phi\left(1 + \mathrm{tn}\ \varepsilon''\ \frac{\mathrm{tn}\ K\Phi}{K}\right)$$

$$g'' \equiv f\cos K\Phi\left(1 + \mathrm{tn}\ \varepsilon'\ \frac{\mathrm{tn}\ K\Phi}{K}\right)$$

f, g' and g'' are also related by

$$f^2 - g'g'' = af. \tag{19'}$$

In addition is it convenient to introduce

$$p'' \equiv \frac{1}{K^2}\left[K\sin K\Phi + (1 - \cos K\Phi)\ \mathrm{tn}\ \varepsilon''\right]$$

$$q \equiv \frac{a}{K^2}(1 - \cos K\Phi). \tag{20}$$

Then using equations (19) and (20), (17) can be rewritten

$$y'' = b'\frac{g'' - x''}{f} + \frac{\alpha'}{f}\left[(x'' - g'')(\ell' - g') - f^2\right] + \delta K^2\left[x''p'' + q\right] \tag{21}$$

Hence the angular deviation is

$$\alpha'' \cong \frac{dy''}{dx''} = -\frac{b'}{f} + \alpha'\frac{\ell' - g'}{f} + \delta K^2 p'' \tag{22}$$

and the object and image points are related by

$$(\ell'' - g'')(\ell' - g') = f^2. \tag{23}$$

When $\ell' = g'$, $\ell'' = \infty$ and when $\ell'' = g''$, $\ell' = \infty$ so g' and g'' are the abscissas of the first and second focal planes G' and G'', respectively.

The principal planes H' and H'' are defined by the property that, for $\delta = 0$, they form images in each other without magnification. If their abscissas are called h' and h'', then $\ell' = h'$, $\ell'' = h''$, $b' = b''$, so from (21) and (23)

$$b' = b'' = b'\frac{g'' - h''}{f}, \qquad (h'' - g'')(h' - g') = f^2$$

so that

$$g'' - h'' = g' - h' = f. \tag{24}$$

Since f is the distance between each focal plane and the corresponding principal plane it can be identified as the focal length of the lens.

With the help of (24) and (25) it can be shown that

$$\frac{1}{\ell' - h'} + \frac{1}{\ell'' - h''} = \frac{1}{f}. \tag{25}$$

The ordinate of the image, b", is given by (21) as

$$b'' = b' \frac{g'' - \ell''}{f} + \delta K^2(\ell'' p'' + q). \tag{26}$$

Since the rays are reversable all of the foregoing equations are valid if primed and double primed variables are interchanged. In particular

$$b' = b'' \frac{g' - \ell'}{f} + \delta K^2(\ell' p' + q). \qquad (26)'$$

Hence all particles will be focussed at ℓ'', 0 for which the relation between b' and δ is

$$b' = \delta K^2(\ell' p' + q) \qquad (26)''$$

which is the condition for velocity-mass focussing.

In order to have the emerging beam entirely parallel to the reference ray it is seen from (22) that

$$\ell' = g', \qquad \frac{b'}{f} = \delta K^2 p''. \tag{27}$$

If these conditions are satisfied the ordinate of the emergent parallel ray is, from (21)

$$y'' = -\alpha' f + \delta K^2 \left[g'' p'' + q\right]. \tag{28}$$

5. Focussing Properties of the System in the Vertical Plane

A description will now be sought for the behavior of an ion which approaches the field at a slight angle α_V' with the horizontal reference plane, and which is displaced from that plane a distance small compared with the dimensions of the system. Such an ion will be deflected vertically only at the edges of the magnetic field by the magnetic fringing field. Since the deflections are proportional to α_V' and $\frac{b_V'}{a}$, where b_V' is the vertical distance above the horizontal plane of the vertical source, the first order horizontal deviations and mass and velocity deviations may be neglected in a first order theory. The situation is shown schematically in Fig. 3.

Fig. 3 is a vertical cross section of the system along the curve $x'0'0''x''$ of Fig. 2. It is shown in Appendix II that an ion which passes through the point $x' = \ell_v'$, $z' = b_v'$ at an angle α_v', i.e., an ion whose vertical path in region I is

$$z = b_v' + (x' - \ell_v')\,\alpha_v' \tag{14v}$$

will be so deflected by the fringing fields that its path in region II is

$$z'' = b_v' \frac{g_v'' - x''}{f_v} + \frac{\alpha_v'}{f_v}\left[(x'' - g_v'')(\ell_v' - g_v') - f_v^2\right] \tag{21v}$$

where

$$f_v = \frac{a\,K_v}{\tan\varepsilon' + \tan\varepsilon'' - \frac{\Phi}{K_v}\tan\varepsilon'\tan\varepsilon''}$$

$$g_v' = f_v\left(1 - \frac{\Phi}{K_v}\tan\varepsilon''\right) \tag{19v}$$

$$g_v'' = f_v\left(1 - \frac{\Phi}{K_v}\tan\varepsilon'\right)$$

$$K_v = \frac{a_m}{a}.$$

A useful relation is

$$f_v^2 - g_v'\,g_v'' = a\,\Phi\,f_v. \tag{19v'}$$

Since (21v) is identical in form to (21) except for the absence velocity-mass dependence, equations (22) to (28) are valid in the vertical plane if subscripts v are placed on all variables and δ_v is set identically equal to zero. Thus

$$\alpha_v'' = -\frac{b_v'}{f_v} + \alpha_v' \frac{\ell_v' - g_v'}{f_v} \tag{22v}$$

$$(\ell_v'' - g_v'')(\ell_v' - g_v') = f_v^2 \tag{23v}$$

$$\frac{1}{\ell_v' - h_v'} + \frac{1}{\ell_v'' - h_v''} = \frac{1}{f_v} \tag{25v}$$

where h_v' and h_v'' are the abscissas of the first and second vertical principal planes, respectively, defined by

$$g_V'' - h_V'' = g_V' - h_V' = f_V. \tag{24v}$$

Again ℓ_V'' is the abscissa of the vertical image whose height is related to that of the object by

$$\frac{b_V''}{b_V'} = \frac{g_V'' - \ell_V''}{f_V} = \frac{f_V}{g_V' - \ell_V'}. \tag{26v}$$

If the beam is to emerge parallel to the horizontal plane it is necessary that

$$\ell_V' = g_V', \qquad b_V' = 0 \tag{27v}$$

in which case the height of an ion which starts with angle α_V' will be

$$z'' = - \alpha_V' f_V. \tag{28v}$$

6. Compound Systems

Suppose there are two systems in series as shown schematically in Fig. 4. The beam is assumed to approach from the left.

The $\left\{\begin{matrix} x_1' \text{ and } y_1' \\ x_2'' \quad\quad y_2'' \end{matrix}\right\}$ axes are the $\left\{\begin{matrix} x' \text{ and } y' \\ x'' \quad\quad y'' \end{matrix}\right\}$ axes for the entire system.

Note that in Fig. 4, ε_2'' is negative. To obtain the path of the emergent ion which passes through ℓ', b' at angle α' equations (21) and (22) are used to calculate the ordinate and angle of the ion when it crosses the first focal plane of system 2. We have

$$x_2' = g_2', \qquad x_1'' = d - g_2'$$

so using (21) and (22)

$$\begin{aligned} b_2' = y_1'' = b' \frac{g_1'' - (d - g_2')}{f_1} + \frac{\alpha'}{f_1} \left[(d - g_2' - g_1'')(\ell' - g_1') - f_1^2\right] \\ + \delta_1 K_1^2 \left[(d - g_2') p_1'' + q_1\right] \end{aligned} \tag{29}$$

$$- \alpha_2' = \alpha_1'' = - \frac{b'}{f_1} + \alpha' \frac{\ell' - g_1'}{f_1} + \delta_1 K_1^2 p_1'' \tag{30}$$

from (21) the equation of the outgoing beam is

$$y'' = b_2' \frac{g_2'' - x''}{f_2} - \alpha_2' f_2 + \delta_2 K_2^2 \left[x'' p_2'' + q_2\right]. \tag{31}$$

Substituting (29) and (30) in (31) gives

$$y'' = b'\left\{\frac{(g_1'' + g_2' - d)}{f_1}\,\frac{(g_2'' - x'')}{f_2} - \frac{f_2}{f_1}\right\}$$
$$+ \frac{\alpha'}{f_1}\left\{\left[(d - g_2' - g_1'')(\ell' - g_1') - f_1^2\right]\frac{(g_2'' - x'')}{f_2} + \frac{\ell' - g_1'}{f_1}\,f_2\right\}$$
$$+ \delta_1 K_1^2\left\{\left[(d - g_2')\,p_1'' + q_1\right]\frac{g_2'' - x''}{f_2} + p_1'' f_2\right\} + \delta_2 K_2^2\left\{x''\,p_2'' + q_2\right\}. \qquad (32)$$

This may be put in the form of (21), which can be rewritten, using (9):

$$y'' = b'\,\frac{G'' - x''}{F} + \frac{\alpha'}{F}\left[(x'' - G'')(\ell' - G') - F^2\right] + (\gamma + B\lambda)\left[x''\,P'' + Q\right] \qquad (21c)$$

by means of the following substitutions:

$$F = \frac{f_1 f_2}{g_1'' + g_2' - d}$$

$$G' = g_1' - \frac{f_1}{f_2}\,F \qquad (19c)$$

$$G'' = g_2'' - \frac{f_2}{f_1}\,F$$

and

$$P'' = -\frac{d - g_2'}{f_2}\,p_1'' + p_2'' - \frac{1}{f_2}\,q_1$$

$$(20c)$$

$$Q = \left[(d - g_2')\,\frac{g_2''}{f_2} + f_2\right]p_1'' + \frac{g_2''}{f_2}\,q_1 + q_2$$

while B is to be construed as an operator which changes p_1'', p_2'', q_1, q_2 into $B_1\,p_1''$, $B_2\,p_2''$, $B_1\,q_1$, $B_2\,q_2$ respectively. The B's are defined by (9).

With these definitions, (21c) describes the path of the outgoing ion and equations (22) through (28) are valid for the system as a whole where, of course, $K^2\delta = \gamma + B\lambda$.

In case the systems are oriented with curvatures opposed, that is, so that the y_1'' and y_2' axes are opposite in sense from each other, the signs of b_2' and α_2' in (29) and (30) are reversed. The effect is to change every sign in (32) except the last: $\delta_2 K_2^2 \left\{x'' p_2'' + q_2\right\}$. This is equivalent to replacing f_2 by $-f_2$ in equations (19c) and (20c).

Similarly equations (19c), (20c), and (21c) describe the vertical focussing, except that there is no first order dependence on γ or λ, so in equation (21c) the term $(\gamma + B\lambda)\left[x'' P'' + Q\right]$ should be omitted. Naturally the vertical f's and g's defined in (19v) must be used in (19c).

The foregoing may be generalized to the case in which the two systems are rotated relative to each other. Thus suppose the y_2' axis in Fig. 4 is rotated about the x_2' axis out of the plane of the paper through an angle θ. Then

$$y_2' = y_1'' \cos\theta + z_1'' \sin\theta$$

$$z_2' = -\,y_1'' \sin\theta + z_1'' \cos\theta$$

if the notation is changed to

$$y_1'' = y_1 \qquad z_1'' = y_2$$

$$y_2' = \bar{y}_1 \qquad z_2' = \bar{y}_2$$

$$C_{11} = \cos\theta \qquad C_{12} = \sin\theta$$

$$C_{21} = -\sin\theta \qquad C_{22} = \cos\theta$$

Then

$$\bar{y}_m = \sum_j C_{mj}\, y_j.$$

The value of y_m when the beam crosses the first focal plane of "2" for the m direction, that is where $x_2' = g_{2m}'$, $x_1'' = d - g_{2m}'$, is

$$\begin{aligned} b_{2m}' &= \sum_j C_{mj}\, y_j(d - g_{2m}') \\ &= \sum_j C_{mj} \left\{ b_j' \frac{g_{1j}'' - (d - g_{2m}')}{f_{1j}} + \frac{\alpha_j'}{f_{1j}} \left[(d - g_{2m}' - g_{1j}'') - f_{1j}^2\right] \right. \\ &\qquad \left. + \delta_{1j}\,(\gamma + B_1\lambda)\left[(d - g_{2m}')\, p_1'' + q_1\right] \right\}. \end{aligned} \qquad (29')$$

Its inclination in the m direction is

$$\alpha'_{2m} = \frac{d\bar{y}_m}{dx'_2} = -\frac{d\bar{y}_m}{dx''_1} = -C_{mj}\frac{dy_j}{dx''_1} = -C_{mj}\,\alpha''_{1j}$$

$$= -\sum_j C_{mj}\left[-\frac{b'_j}{f_{1j}} + \alpha'_j\,\frac{\ell'_j - g'_{1j}}{f_{1j}} + \delta_{1j}\,(\gamma + B_1\,\lambda)\,p''_1\right]. \tag{30'}$$

In (29') and (30') the first index on the f's and g's refers to system "1" or "2," the second to the horizontal or vertical component. The outgoing ray has its m^{th} ordinate given by

$$y''_m = b'_{2m}\,\frac{g''_{2m} - x''}{f_{2m}} - \alpha'_{2m}\,f_{2m} + \delta_{1m}\,(\gamma + B_2\,\lambda)\left[x''\,p''_2 + q_2\right] \tag{31'}$$

substituting (29') and (30') in (31') gives

$$y''_m = \sum_{j=1}^{2} C_{mj}\left\{b'_j\,\frac{G''_{mj} - x''}{F_{mj}} + \frac{\alpha'_j}{F_{mj}}\left[(x'' - G'_{mj})(\ell'_j - G'_{mj}) - F^2_{mj}\right]\right\}$$
$$+ (\gamma + B\,\lambda)\left[x''\,P''_m + Q_m\right] \tag{21c'}$$

where

$$F_{mj} = \frac{f_{1j}\,f_{2m}}{g''_{1j} + g'_{2m} - d}$$

$$G'_{mj} = g'_{1j} - \frac{f_{1j}}{f_{2m}}\,F_{mj} \tag{19c'}$$

$$G''_{mj} = g''_{2m} - \frac{f_{2m}}{f_{1j}}\,F_{mj}.$$

Here F_{mj} refers to the outgoing m^{th} component due to incoming j^{th} component. Also

$$P''_m = -C_{m1}\,(d - g'_{2m})\,\frac{p''_1}{f_{2m}} + \delta_{1m}\,p'_2 - \frac{C_{m1}}{f_{2m}}\,q_1$$

$$Q_m = C_{m1}\left[(d - g'_{2m})\,\frac{g''_{2m}}{f_{2m}} + f_{2m}\right]p''_1 + C_{m1}\,\frac{g''_{2m}}{f_{2m}}\,q_1 + \delta_{1m}\,q_2 \tag{20c'}$$

and B has the same meaning as before.

7. Applications to Bevatron Injector Systems

a. The Berkeley 1/4 scale model bevatron.

The injector system for this machine (see Fig. 5) consists of a cycltron which accelerates the protons to 0.625 Mev, ejecting them in a beam which appears to have separate horizontal and vertical nodes, about 2 ft. from the cyclotron tank wall, the horizontal node being about 5 in. closer to the cyclotron than the vertical node. The beam then passes through a magnetic wedge with a 10 in. radius of curvature, by which it is directed to a 90° electrostatic deflector which brings the beam into the bevatron.

How should this compound system of magnet and deflector be designed to obtain a parallel beam for injection, with optimum definition? In the notation of Sec. 6, system 1 is the magnetic wedge; 2 the electrostatic deflector. They are distinguished by subscripts M and E respectively. As before, subscript v refers to the vertical plane, h to the horizontal. Since the velocity of the protons is non-relativistic, and since there is no mass variation the following relations hold.

$$
\begin{array}{llll}
K_{Mh} = K_{Mv} = 1 & K_{Ev} = \infty & K_{Eh} = \sqrt{2} & f_{Eh} = \dfrac{a_E}{\sqrt{2}\sin\sqrt{2}\,\Phi_E} \\
B_M = 1 & B_E = 2 & & g'_{Eh} = g''_{Eh} \equiv g_E = \dfrac{a_E}{\sqrt{2}}\cot\sqrt{2}\,\Phi_E \\
K^2_{Mh}\delta = \lambda & K^2_{Eh}\delta = 2\lambda & & P_E = \dfrac{1}{\sqrt{2}}\sin\sqrt{2}\,\Phi_E \\
\ell'_h - \ell'_v = 5'' & \Phi_E = \pi/2 & \varepsilon'_E = \varepsilon''_E = 0 & q_E = \dfrac{a_E}{2}(1 - \cos\sqrt{2}\,\Phi_E).
\end{array}
\tag{32}
$$

If the value $\Phi_E = \pi/2$, then

$$
f_E = .890\ a_E \qquad p_E = .563
$$

$$
g_E = -.539\ a_E \qquad q_E = .197\ a_E.
$$

It is assumed that a_E and a_M are fixed, so that ℓ'_h (or ℓ'_v) ε' ε'', Φ_M, and d remain to be determined.

Now $b'_h = b'_v = 0$ and there is no velocity dependence in the vertical plane, so the conditions for a parallel beam as given by (27) are

$$\ell'_h = G'_h \tag{33a}$$

$$\delta\, K^2 P'' = \lambda\, BP'' = 0 \tag{33b}$$

$$\ell'_v = G'_v. \tag{33c}$$

Equations (33) determine three of the parameters in terms of the other two; for convenience they will be used to express ε', ε'', and d in terms of ℓ_h' and Φ_M.

From (19c), (33a) becomes

$$\ell_h' = g_M' - \frac{f_M^2}{g_M'' + (g_E - d)}$$

where the subscript h on the f's and g's is understood, or

$$(\ell_h' - g_M')\Big((d - g_E) - g_M''\Big) = f_M^2$$

a comparison with (23) shows that $d - g_E$ is the image of ℓ_h' by the magnetic wedge alone. Hence (18) applies with $\ell_h'' = d - g_E$, $K = 1$, $\Phi = \Phi_M$:

$$\frac{1}{d - g_E} = -\frac{\tan\,\varepsilon''}{a_M} + \frac{1}{a_M}\,\frac{\ell_h'\,(\tan\,\Phi_M - \tan\,\varepsilon') - a_M}{\ell_h'\,(1 + \tan\,\varepsilon'\,\tan\,\Phi_M) + a_M \tan\,\Phi_M}$$

or

$$\tan\,\varepsilon'' = \frac{\ell_h'\,(\tan\,\Phi_M - \tan\,\varepsilon') - a_M}{\ell_h'\,(1 + \tan\,\varepsilon'\,\tan\,\Phi_M) + a_M\,\tan\,\Phi_M} - \frac{a_M}{d - g_E}. \tag{34a}$$

From (20c) equation (33b) can be written as follows:

$$BP'' = -\frac{d - g_E}{f_E}\,B_M\,p_M'' + B_E\,p_E'' - \frac{B_M\,q_M}{f_E} = 0 \tag{34b}$$

$$d - g_E = \frac{2\,p_E''\,f_E - q_M}{p_M''} = \frac{a_E - a_M\,(1 - \cos\,\Phi_M)}{\sin\,\Phi_M + (1 - \cos\,\Phi_M)\,\tan\,\varepsilon''} \tag{34b'}$$

substituting this in (34a) gives

$$\tan\,\varepsilon''\left(1 + \frac{1 - \cos\,\Phi_M}{\frac{a_E}{a_M} - (1 - \cos\,\Phi_M)}\right) = \frac{\ell_h'\,(\tan\,\Phi_M - \tan\,\varepsilon') - a_M}{\ell_h'\,(1 + \tan\,\varepsilon'\,\tan\,\Phi_M) + a_M\,\tan\,\Phi_M} - \frac{\sin\,\Phi_M}{\frac{a_E}{a_M} - (1 - \cos\,\Phi_M)}$$

$$\tan\,\varepsilon'' = \left[1 - \frac{a_M}{a_E}\,(1 - \cos\,\Phi_M)\right]\frac{\ell_h'\,(\tan\,\Phi_M - \tan\,\varepsilon') - a_M}{\ell_h'\,(1 + \tan\,\varepsilon'\,\tan\,\Phi) + a_M\,\tan\,\Phi_M} - \frac{a_M}{a_E}\,\sin\,\Phi_M. \tag{34ab}$$

Since all of the vertical focussing takes place in the magnetic wedge, $g_V' = g_{MV}'$ so from (19v)

$$\ell_v' = \frac{a_M\ (1 - \Phi_M \tan \varepsilon'')}{\tan \varepsilon'' + \tan \varepsilon'\ (1 - \Phi \tan \varepsilon'')}$$

or

$$\tan \varepsilon'' = \frac{a_M - \ell_v' \tan \varepsilon'}{\ell_v'\ (1 - \Phi_M \tan \varepsilon') + a_M\ \Phi_M}. \tag{34c}$$

Setting

$$A_E = \frac{a_E}{a_M}, \qquad L_h' = \frac{\ell'}{a_M}, \qquad L_v' = \frac{\ell_v'}{a_M} \tag{35}$$

and equating (34c) and (34ab) it follows that

$$\frac{1 - L_v' \tan \varepsilon'}{L_v'\ (1 - \Phi_M \tan \varepsilon') + \Phi_M} = \left[1 - \frac{1}{A_E}\ (1 - \cos \Phi_M)\right] \frac{L_h'\ (\tan \Phi_M - \tan \varepsilon') - 1}{L_h'\ (1 + \tan \varepsilon' \tan \Phi_M) + \tan \Phi_M} - \frac{\sin \Phi_M}{A_E}. \tag{36}$$

Let μ and ν be defined by the equations

$$\frac{1}{\mu} = 1 - \frac{1}{A_E}\ (1 - \cos \Phi_M)$$
$$\frac{1}{\nu} = \frac{\sin \Phi_M}{A_E}. \tag{37}$$

(In case the y_M'' and y_E' axes are antiparallel, A_E should be replaced by $-A_E$ in (34b) and (37)). With these definitions (36) may be written

$$\begin{aligned}
&\tan^2 \varepsilon' \left[\tan \Phi_M + \frac{\Phi_M}{\mu} + \frac{\Phi_M \tan \Phi_M}{\nu}\right] \\
&+ \tan \varepsilon' \left[\left(\frac{1}{\mu L_v'} + \frac{1}{\mu L_h'} + \frac{1}{\nu}\right)\Phi_M + \left(\frac{1}{L_h'} - \frac{1}{L_v'} - \frac{1}{\nu}\right)\tan \Phi_M\right. \\
&\qquad \left. + \left(-\frac{1}{\mu} + \frac{1}{\nu L_h'} - \frac{1}{\nu L_v'}\right)\Phi_M \tan \Phi_M + \left(1 - \frac{1}{\mu}\right)\right] \\
&+ \left[\left(\frac{1}{\mu L_h' L_v'} - \frac{1}{\nu L_v'}\right)\Phi_M + \left(-\frac{1}{L_h' L_v'} + \frac{1}{\mu} - \frac{1}{\nu L_h'}\right)\tan \Phi_M\right. \\
&\qquad \left. + \left(\frac{1}{\nu L_v'} - \frac{1}{\nu L_h' L_v'}\right)\Phi_M \tan \Phi_M - \left(\frac{1}{L_v'} - \frac{1}{\mu L_h'} + \frac{1}{\nu}\right)\right] = 0.
\end{aligned} \tag{36'}$$

From this equation ε' can be expressed in terms of ℓ_h', ℓ_v' and Φ_M. Then, substituting back in (34c) and (34b'), ε'' and d may be similarily expressed. ℓ_h', ℓ_v'

and Φ_M remain to be disposed of and will be chosen to give as narrow a beam as possible.

The horizontal and vertical widths of the beam may be determined as follows. From (28) the horizontal width due to angular divergence is*

$$\Delta_{\alpha_h} = -\alpha' F$$

From (19c) and (33a) $F = \frac{f_E}{f_M}(G_h' - g_{Mh}') = \frac{f_E}{f_M}(\ell_h' - g_{Mh}')$, so

$$\Delta_{\alpha_h} = \frac{\alpha' f_E}{(d - g_E) f_M}\left[(\ell' - g_M')\left((d - g_E) - g_M''\right) + \ell' g_M'' - g_M' g_M''\right]$$

$$= \frac{\alpha' f_E}{(d - g_E) f_M}\left[f_M^2 - g_M' g_M'' + \ell' g_M''\right] = \frac{\alpha' f_E}{(d - g_E) f_M}\left[a_M f_M + \ell' g_M''\right].$$

Thus

$$\Delta_{\alpha_h} = \alpha_h' \frac{f_E}{d - g_E}\left[a_M + \ell' \cos\Phi_M(1 + \tan\varepsilon' \tan\Phi_M)\right] \qquad (38a)$$

while that due to velocity deviation is

$$\Delta_\lambda = \delta K^2\left[G'' P'' + Q\right] = \lambda B\left[G'' P'' + Q\right] = \lambda BQ$$

since $BP'' = 0$ by (33b)

$$\Delta_\lambda = \lambda\left\{\left[(d - g_E)\frac{g_E}{f_E} + f_E\right] p_M'' + \frac{g_E}{f_E} q_M + 2q_E\right\}$$

with the help of (34b) this can be written

$$\Delta_\lambda = \lambda\left\{f_E\, p_M'' + 2q_E + 2g_E\, p_E''\right\}$$

$$\Delta_\lambda = \lambda\left\{f_E\left[\sin\Phi_M + (1 - \cos\Phi_M)\tan\varepsilon''\right] + a_E\right\}. \qquad (38b)$$

Finally the vertical height is given by (28v)

$$\Delta_{\alpha_v} = -\alpha_v' f_v = -\alpha_v' \frac{g_{Mv}'}{1 - \Phi_M \tan\varepsilon''}$$

but since $\ell_v' = g_v' = g_{Mv}'$

$$\Delta_{\alpha_v} = -\alpha_v' \frac{\ell_v'}{1 - \Phi_M \tan\varepsilon''} \qquad (38c)$$

Formulas (38) give the widths of the beam. ℓ_h' and Φ_M (and hence ℓ_v', ε', ε'', and d) should be chosen to make them as small as possible.

* If subscripts v or h are omitted from appropriate variables, h is understood.

For a numerical example let

$$a_M = 10'', \qquad a_E = 25'', \qquad \Phi_E = 90^\circ.$$

Then the optimum values are approximately $\ell_h' = 35''$, $\Phi_M = 19^\circ$.

Application of the above recipe then yields

$$\ell_v' = 30'', \qquad \varepsilon' = 40.^\circ0, \qquad \varepsilon'' = -35.^\circ7, \qquad d = 72.5'' \text{ and}$$

$$\Delta_{\alpha_h} = .238\ \alpha_h' \ \frac{\text{inches}}{\text{degree}}$$

$$\Delta_\lambda = .307\ \lambda\ \frac{\text{inches}}{\text{percent}}$$

$$\Delta_{\alpha_v} = .411\ \alpha_v' \ \frac{\text{inches}}{\text{degree}}.$$

b. The full scale Berkeley bevatron.

For this machine it is proposed to use a linear accelerator rather than a cyclotron in the injection system, since the former gives a beam of much smaller width and angular divergence than the latter. The problem then is to bring the beam from the linear accelerator into the bevatron without spoiling this good definition. Suppose a single electrostatic deflector is used. Then from (21) and (22)

$$y_h'' = b_h' \frac{g_E - x''}{f_E} + \frac{\alpha_h'}{f_E}\left[(x'' - g_E)(\ell_h' - g_E) - f_E^2\right] + 2\lambda\left[x'' p_E + q_E\right]$$

$$\alpha'' = -\frac{b_h'}{f_E} + \alpha_h' \frac{\ell_h' - g_E}{f_E} + 2p_E\lambda.$$

Now if $\sqrt{2}\,\Phi_E < \pi/2$, then $g_E > 0$ and ℓ_h' may be taken equal to g_E, in which case these equations reduce to

$$y'' = b_h' \frac{g_E - x''}{f_E} - \alpha_h' f_E + 2\lambda\,(x'' p_E + q_E)$$

$$\alpha'' = -\frac{b_h'}{f_E} + 2p_E\lambda, \tag{39}$$

while in the vertical direction,

$$y''_v = b'_v - \alpha'_v \,(g_E + a_E\,\Phi_E) \qquad (39v)$$

$$\alpha''_v = -\,\alpha'_v$$

since there is no vertical deflection by the electric field.

It is estimated that the linear accelerator will give a beam about 1/4 in. in diameter, with an angular divergence of about 1:500, and an energy spread of about 1:300, this means

$$\Delta b' \sim 1/4'', \qquad \Delta\alpha' \sim \frac{1}{500}, \qquad \Delta\lambda \sim \frac{1}{500}.$$

The figures tentatively adopted for the deflector are

$$\Phi_E = 37^{\circ}\!.5 \qquad a_E = 20'.$$

Thus at the end of the deflector the injected beam would be given by (39) and (39v) as

$$y'_h = .6017\, b'_h - 212.4\, \alpha'_h + 93.92\,\lambda \qquad y''_v = b'_v - 284.9\, \alpha'_v$$

$$\alpha''_h = -\frac{b'_h}{212.4} + 1.130\,\lambda \qquad \alpha''_v = -\,\alpha'_v.$$

The above estimates for the b's and α's give

$$\Delta y''_h \sim .1504 + .4248 + .1878 \sim .8'' \qquad \Delta y''_v \sim .25 + .57 \sim .8''$$

$$\Delta\alpha''_h \sim .0012 + .0023 \sim .0035 \text{ rad} \qquad \Delta\alpha''_v = \Delta\alpha'_v \sim .002.$$

This should be sufficiently good definition for the purpose.

Appendix A

Path of the Ions Inside the Field Region, Horizontal

The equation of motion in region III is

$$\frac{d}{dt}(m\vec{v}) = e\vec{E} + e\,\frac{\vec{v} \times \vec{H}}{c}$$

where m is the relativistic mass of the ion. Using polar coordinates r, φ of Fig. 1 and the fields described in Sec. 2 there result from this vector equation the two scaler equations of motion

$$\frac{d}{dt}(m\, r^2 \dot{\varphi}) = \frac{eH}{c}\, r\, \dot{r} \tag{A1}$$

$$\frac{d}{dt}(m\, \dot{r}) - m\, r\, \dot{\varphi}^2 = -\frac{1}{r}\,\frac{e\,X}{\ln\frac{R_1}{R_2}} - \frac{eH}{c}\, r\, \dot{\varphi}. \tag{A2}$$

Integrating (A1) gives

$$\dot{\varphi} = \frac{m_1\, r_1^2}{mr^2}\,\dot{\varphi}_1 + \frac{eH}{2mc}\left(1 - \frac{r_1^2}{r^2}\right)$$

where subscript 1 denotes the value of the quantity at $\varphi = 0$. Putting

$$\begin{aligned} r &= a\,(1+\rho) \\ \rho_1 &= \frac{y_1}{a} \end{aligned} \tag{A3}$$

gives, to first order in ρ and ρ_1

$$\begin{aligned} \dot{\varphi} &= \frac{m_1}{m}\dot{\varphi}_1\,(1 + 2\rho_1 - 2\rho) + \frac{eH}{mc}(\rho - \rho_1) \\ \dot{\varphi}^2 &= \left(\frac{m_1}{m}\dot{\varphi}_1\right)^2 (1 + 4\rho_1 - 4\rho) + \frac{eH}{c}(\rho - \rho_1). \end{aligned} \tag{A4}$$

Substituting (A4) in (A2) gives

$$\begin{aligned} &\frac{d}{dt}(m\,\dot{\rho}) - \frac{m_1\,\dot{\varphi}_1}{m}\left[m_1\,\dot{\varphi}_1\,(1 + 4\rho_1 - 3\rho) + \frac{2eH}{c}(\rho - \rho_1)\right] \\ &= -\frac{e\,X}{a^2 \ln\frac{R_1}{R_2}}(1-\rho) - \frac{eH}{c}\frac{1}{m}\left[m_1\,\dot{\varphi}_1\,(1 + 2\rho_1 - \rho) + \frac{eH}{c}(\rho - \rho_1)\right]. \end{aligned} \tag{A5}$$

Now from (4) the velocity* and rest mass in region I

$$\beta = \frac{v}{c} = \beta_o\ (1 + \lambda) \qquad (4)$$

$$M = M_o\ (1 + \gamma).$$

Let E be the total self energy (kinetic and rest) of the particle, and **p** its linear momentum. By conservation of energy $E_{III}(r) = E_I - eV\ (r)$, or

$$m(r) = \frac{E_{III}(r)}{c^2} = m_I - \frac{eV(r)}{c^2}.$$

Now

$$m_I = \frac{M_o\ (1 + \gamma)}{\sqrt{1 - \beta_o^2\ (1 + \lambda)^2}} \cong \frac{M_o}{\sqrt{1 - \beta_o^2}} \left(1 + \gamma + \frac{\beta_o^2}{1 - \beta_o^2}\right)$$

while from (1) and (3)

$$\frac{eV(r)}{c^2} = \frac{a}{a_e} \frac{M_o\ \beta_o^2}{\sqrt{1 - \beta_o^2}}$$

so to first order in γ, λ, and ρ

$$m(\rho) = \frac{M_o}{\sqrt{1 - \beta_o^2}} \left[1 + \gamma + \frac{\beta_o^2}{1 - \beta_o^2}\ \lambda - \frac{a}{a_e}\ \beta_o^2\ \rho\right]. \qquad (A6)$$

Now $\beta = \sqrt{1 - \frac{M^2}{m^2}}$ whence $p = m\beta c = c\ \sqrt{m^2 - M^2}$

so that from (4) and (A6)

$$p(\rho) = \frac{M_o\ \beta_o\ c}{\sqrt{1 - \beta_o^2}} \left(1 + \gamma + \frac{\lambda}{1 - \beta_o^2} - \frac{a}{a_e}\ \rho\right). \qquad (A7)$$

Combining (A6) and (A7) gives $v = p/m$:

$$v(\rho) = \beta_o c \left[1 + \lambda - \frac{a}{a_e}\ (1 - \beta_o^2)\ \rho\right]. \qquad (A8)$$

Further $r_1\dot{\varphi}_1 = v_1 \cos \alpha' \approx v_1$

---

* It is sometimes more useful to use the energy rather than the velocity. It may be shown from (4) and (A6) that the kinetic energy in region I

$$T = T_o \left[1 + \gamma + \frac{1 + \sqrt{1 - \beta_o^2}}{1 - \beta_o^2}\ \lambda\ \right].$$

$$m_1\,\varphi_1 \cong \frac{m_1\,v_1}{r_1} = \frac{p_1}{a}(1-\rho_1) = \frac{M_o\,\beta_o\,c}{a\sqrt{1-\beta_o^2}}\left[1+\gamma+\frac{\lambda}{1-\beta_o^2} - y_1\left(\frac{1}{a}+\frac{1}{a_e}\right)\right]. \qquad \text{(A9)}$$

The equation of motion (A5) is now multiplied by $m\,\dfrac{1-\beta_o^2}{M_o^2}$ and (A6) and (A9) are substituted into it. By noting that

$$\dot{\rho} \approx \frac{v}{a}\,\alpha$$

$$\ddot{\rho} \approx \frac{v^2}{a^2}\frac{d\alpha}{d\varphi} = 0\left(\frac{v^2}{a^2}\frac{\alpha}{\Phi}\right) = 0\left(\frac{v^2}{a^2}\,\alpha\right)$$

$$(\dot{\rho})^2 \cong \frac{v^2}{a^2}\,\alpha^2$$

so that $(\dot{\rho})^2$ is second order, compared to $\ddot{\rho}$; keeping only first order terms, and making use of (3) to replace the field expressions by a_m and a_e, and of (2') to simplify many terms, the following equation of motion is obtained:

$$\ddot{\rho} = \frac{v_o^2}{a^2}\left\{\left[\gamma+\lambda\left(1+\frac{a}{a_e}+\frac{\beta_o^2}{1-\beta_o^2}\right)\right] - \left[1+\left(\frac{a}{a_e}\right)^2 - \left(\frac{a}{a_e}\right)^2\beta_o^2\right]\rho\right\}.$$

By use of the substitutions (8) and (9)

$$K^2 = 1+\left(\frac{a}{a_e}\right)^2(1-\beta_o^2) \qquad (8)$$

$$K^2\delta = \gamma + B\lambda = \gamma+\lambda\left(1+\frac{a}{a_e}+\frac{\beta_o^2}{1-\beta_o^2}\right) \qquad (9)$$

the equation of motion becomes

$$\ddot{\rho} = \frac{v_o^2}{a^2}K^2(\delta-\rho). \qquad \text{(A10)}$$

Integrating (A10) with the boundary conditions

$$\rho_o = \frac{y_1}{a},\quad \left(\frac{d\rho}{dt}\right)_o = -\alpha'\frac{v_o}{a} \quad \text{at } t = 0$$

gives

$$\rho(t) = -\frac{\alpha'}{K}\sin\frac{v_o}{c}Kt + \delta\left(1-\cos\frac{v_o}{a}Kt\right) + \frac{y_1}{a}\cos\frac{v_o}{a}Kt.$$

To zero order $t = \varphi\,\dfrac{a}{v_o}$. Since all terms on the right are already first order, this value of t can be used in the above equation, giving

$$r - a = a\rho = a\left[-\frac{\alpha'}{K}\sin K\varphi + \delta\,(1 - \cos K\varphi) + \frac{y_1}{a}\cos K\varphi\right]. \tag{7}$$

Appendix B

Vertical Focussing (See Fig. 5)

The vertical motion is shown in Fig. 3, and in more detail in Fig. 5. In the latter figure light solid lines are in the yz plane, the plane of the paper. Dashed lines are in the xy plane. As before reference ray approaches the field in the $-x'$ direction. $\vec{n}'$ is the normal to the plane bounding the fields, $\varepsilon' = \angle(x', n')$.

The force on the ion is

$$\vec{F} = e\left[\vec{E} + \frac{1}{c}\vec{v} \times \vec{H}\right] = m\ddot{\vec{r}}$$

so

$$\ddot{z} = \frac{e}{mc}(v_x H_y - v_y H_x) = \frac{e}{mc} v_x H_y, \qquad \text{since } H_x = 0$$

but

$$v_x = -v_y \; \tan\varepsilon'$$

$$\dot{z} = -\frac{e}{mc} \tan\varepsilon' \int H_y(y,z)\frac{dy}{dt}\,dt = -\frac{e}{mc}\tan\varepsilon' \int_{-\ell_V'\cos\varepsilon'}^{y} H_y(y,z)\,dy + \dot{z}_0 .$$

It is desired to know $\dot{z}$ just inside the magnet, past the fringing field. Since $\vec{H}$ is irrotational

$$\oint \vec{H}\cdot d\vec{r} = \underbrace{\int_{-\ell_V'\cos\varepsilon'}^{y} H_y(y,z)\,dy}_{\text{actual path}} + \underbrace{\int_{z_1}^{0} H_z\,dz}_{\text{inside}} + \int_{y}^{-\ell_V'\cos\varepsilon'} H_y(y,0)\,dy = 0.$$

By symmetry $H_y = 0$ on the $z = 0$ plane, so the integral in the expression for $\dot{z}$, at $y \approx 0$ is

$$-\int_{z_1}^{0} H_z\,dz = H\,z_1$$

where z_1 is the height at which the ion enters the wedge. Hence,

$$\dot{z} - \dot{z}_o = - \frac{eH}{mc} \tan \varepsilon' \; z_1.$$

Dividing by v, and using the second of equations (3) gives, to first order*

$$\Delta\gamma' = - \frac{z_1}{a_m} \tan \varepsilon' = - \frac{z_1}{a} \frac{\tan \varepsilon'}{K_V} \tag{B1}$$

where

$$K_V = \frac{a_m}{a}. \tag{B2}$$

If the ion passes through the point $x' = \ell_V'$, $z' = b_V'$ then $z = b_V' + \ell_V' \gamma'$

$$\Delta\gamma' = - \frac{b_V' + \ell' \gamma_V'}{a} \frac{\tan \varepsilon'}{K_V}.$$

The ion now goes a distance $a\Phi$ (1 + small correction) in the field, emerging with

$$z_2 = z_1 + (\gamma' + \Delta\gamma')\, a\Phi + \text{second order terms}$$

$$= b_V' + \ell_V' \gamma' + \left[\gamma' - \left(\frac{b_V'}{a} + \frac{\ell_V'}{a} \gamma'\right) \frac{\tan \varepsilon'}{K_V}\right] a\Phi$$

here it undergoes another deflection $\Delta\gamma'' = - \frac{z_2}{a} \frac{\tan \varepsilon''}{K_V}$ so that the outgoing angle is

$$\alpha_V'' = \frac{dz}{dx''} = \gamma' + \Delta\gamma' + \Delta\gamma''$$

$$\alpha_V'' = - \frac{b_V'}{a\, K_V}\left[\tan \varepsilon' + \tan \varepsilon'' - \frac{\Phi}{K_V} \tan \varepsilon' \tan \varepsilon''\right] \tag{B3}$$

$$- \gamma' \left[\frac{\ell_V'}{a\, K_V}\left(\tan \varepsilon' + \tan \varepsilon'' - \frac{\Phi}{K_V} \tan \varepsilon' \tan \varepsilon''\right) - \left(1 - \frac{\Phi}{K_V} \tan \varepsilon''\right)\right].$$

Hence the vertical path of the ion in region II is given by

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*This derivation has been given by L. S. Lavatelli, AEC Oak Ridge report MDDC-350

$$z'' = z_2 + x'' \alpha_v'' = b_v' \left\{ - \frac{x''}{a\,K_v} \left(\tan\varepsilon' + \tan\varepsilon'' - \frac{\Phi}{K_v} \tan\varepsilon' \tan\varepsilon'' \right) + \left(1 - \frac{\Phi}{K_v} \tan\varepsilon' \right) \right\}$$

$$+ \alpha' \left\{ x'' \left[\frac{l_v'}{a\,K_v} \left(\tan\varepsilon' + \tan\varepsilon'' - \frac{\Phi}{K_v} \tan\varepsilon' \tan\varepsilon'' \right) - \left(1 - \frac{\Phi}{K_v} \tan\varepsilon'' \right) \right] - \left[l_v' \left(1 - \frac{\Phi}{K_v} \tan\varepsilon' \right) - a\Phi \right] \right\} \tag{B4}$$

which goes into the standard form

$$z'' = b_v' \frac{g_v'' - x''}{f_v} + \frac{\alpha_v'}{f_v} \left[(x'' - g_v'')(l_v' - g_v') - f_v^2 \right] \tag{21v}$$

with the definitions

$$f_v = \frac{a\,K_v}{\tan\varepsilon' + \tan\varepsilon'' - \frac{\Phi}{K_v} \tan\varepsilon' \tan\varepsilon''}$$

$$g_v' = f_v \left(1 - \frac{\Phi}{K_v} \tan\varepsilon'' \right) \tag{19v}$$

$$g_v'' = f_v \left(1 - \frac{\Phi}{K_v} \tan\varepsilon' \right)$$

where

$$K_v = \frac{a_m}{a}$$

f_v, g_v' and g_v'' are also related by

$$f_v^2 - g_v' g_v'' = a\Phi f_v . \tag{19v)'$$

Appendix C

<u>Effect of Fringing Fields</u> (See Fig. 6)

As the ion approaches the fields along the x' axis its path is bent somewhat by the fringing fields, which exert a normal force

$$F_n = \frac{m\,v^2}{r} \cong e\,E_x \sin\varepsilon' + e\,E_y \cos\varepsilon' + \frac{He\,v}{c}.$$

But $dx = r d\varphi \cos\varphi \sim r d\varphi \cos\varepsilon'$, where $d\varphi$ is the change in direction in a distance dx

$$\frac{d\varphi}{dx} = \frac{E_x \ln \varepsilon' + E_y}{E_o\ a_E} + \frac{H_z}{H_o\ a_M \cos \varepsilon'}$$

or

$$\Delta\varphi = \frac{1}{E_o\ a_e}\left[\ln\ \varepsilon' \int E_x\ dx + \int E_y\ dx\right] + \frac{1}{H_o\ a_M \cos \varepsilon'} \int H_z\ dx \tag{C1}$$

$$= \frac{\ln\ \varepsilon'\ \Delta V_E + \Delta U_E}{E_o\ a_E} + \frac{\Delta U_M}{H_o\ a_M \cos \varepsilon'}$$

where U_E and U_M are the electric and magnetic stream functions, and V_E is the electric potential function. Whenever an electric field is present, $\varepsilon' = 0$, so the first term is always absent. To calculate the addition to Φ due to the fringing fields the differences in $\Delta\varphi$ are taken with and without fringing fields. Thus in the ideal case, without fringing field $\frac{\Delta U_E}{V_o} = \frac{2x}{R_1 - R_2}$, where the condenser plates are at potentials V_o and $-V_o$. In the real case, the fringing field is given by the field of two sem-infinite conducting planes, at potential V_o and $-V_o$. This is given by Smythe, problem 24, Chapter IV:

$$z = x + iy = \frac{R_1 - R_2}{\pi} \ln \sin \frac{\pi(V_E + i\ U_E)}{2V_o}.$$

With $y = 0$ it is found that

$$\Delta U_E = U_E\ (x \longrightarrow \infty) - U_E\ (x \longrightarrow -\infty) = \frac{2x}{R_1 - R_2} + \frac{V_o}{\pi} \ln 4.$$

Thus the change in Φ due to electric fringing field is

$$\Delta\Phi_E = \frac{2 \ln 4}{\pi} \frac{R_1 - R_2}{a_E} = .883\ \frac{R_1 - R_2}{a_E}. \tag{C2}$$

For the magnetic field, if it is assumed that the magnet extends infinitely in the vertical direction, then it may be shown, by a suitable series of Schwarz transformations, that

$$\pi \frac{Z}{S_v} = \frac{\pi}{S_v}(x + iz) = -\sqrt{1 + 4 \exp. \frac{\pi(U_M + i\ V_M)}{V_{oM}}} + \sinh^{-1} \frac{1}{2} \exp.\left(-\frac{\pi}{2} \frac{U_M + i\ V_M}{V_o}\right)$$

where S_v is the vertical separation of the magnet pole faces. Thus

$$\Delta \frac{U_M}{V_{oM}} = \frac{1}{V_{oM}}\left[U_M\ (x \longrightarrow \infty) - U_M\ (x \longrightarrow -\infty)\right]$$

$$= \frac{2x}{S_v} + \frac{2}{\pi}\left(1 + \ln \frac{\pi}{4}\frac{x'}{S_v}\right)$$

whereas without the fringing field it is $\frac{2x}{S_v}$. Hence

$$\Delta\Phi_M' = \frac{S_v}{\cos \varepsilon' \ a_M}\frac{1}{\pi}\left(1 + \ln \frac{\pi}{4}\frac{x'}{S_v}\right). \tag{C3}$$

Actually the magnet is not vertically infinite, but has height A so that for $x' \gg A$ the contribution to $\Delta\Phi_M$ will be zero. Thus x' should be limited to the approximate dimentsions of the system, namely A,

$$\Delta\Phi_M \cong \frac{S_v}{a_M}\frac{1}{\pi}\left(1 + \ln \frac{\pi}{4}\frac{A}{S_v}\right)(\sec \varepsilon' + \sec \varepsilon'').$$

The exact value is

$$\Delta\Phi_M = \lim_{x \longrightarrow \infty} \frac{1}{a_M}\left\{\sec \varepsilon' \left[\int_{-\infty}^{x} \frac{H_z\ (x, x \tan \varepsilon')}{H_o}\ dx - x\right] + \sec \varepsilon'' \left[\int_{-\infty}^{x} \frac{H_z\ (x, x \tan \varepsilon'')}{H_o}\ dx - x\right]\right\}. \tag{C4}$$

Hence the fringing fields increase the turning angle Φ by

$$\Delta\Phi = \Delta\Phi_E + \Delta\Phi_M \sim .883\ \frac{R_1 - R_2}{a_E} + \frac{S_v}{a_M}\frac{1}{\pi}\ (\sec \varepsilon' + \sec \varepsilon'')\ \ln\left(\frac{\pi e}{4}\frac{A}{S_v}\right) \tag{C5}$$

where S_v is the vertical separation of the pole faces and A is of the order of the total height of the magnet. Thus, due to the fringing fields, both the electric and magnetic fields effectively extend an additional distance of the order of the gap width.

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Errata

Page

4 The term "electromagnetic field" refers to the static electric and magnetic fields discussed in this paper.

Following the first sentence after equation (1) add:

> The x' and x" axes of Fig. 1 are normal to y' and y" at O' and O" respectively, directed outward, so that the ion just described approaches the field along the x' axis negatively and leaves along the x" axis positively.

15 After equation (32) read:

> If the value $\phi_E = \pi/2$ is substituted in (32) the following numerical values are obtained:

21 The last term on the right of the second of equation (A4) should read:

$$\frac{eH}{mc}(\rho - \rho_1) \cdot \frac{m_1}{m}\dot{\varphi}_1 \quad \text{not} \quad \frac{eH}{c}(\rho - \rho_1)$$

27 In the first equation in the middle of the page y is measured from the inner condenser plate, at potential $-V_o$.

The following equation is valid for $y = \frac{R_1 - R_2}{2}$, that is for $V_E = 0$, rather than for $y = 0$. It should read:

$$\Delta U_E = V_o\left[\frac{2x}{R_1 - R_2} + \frac{1}{\pi}\ln 4\right]$$

Figure

5 The separation between magnet pole faces should be labeled S_V, not S_H.

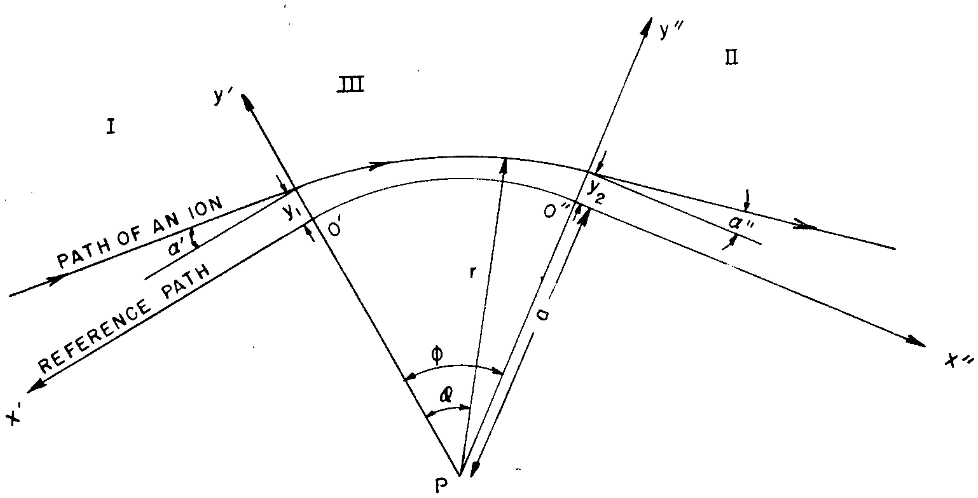
y″
II
y′
III
I
PATH OF AN ION
REFERENCE PATH
y_1
O′
y_2
O″
a′
a″
r
a
φ
α
x′
x″
P

Fig. 1

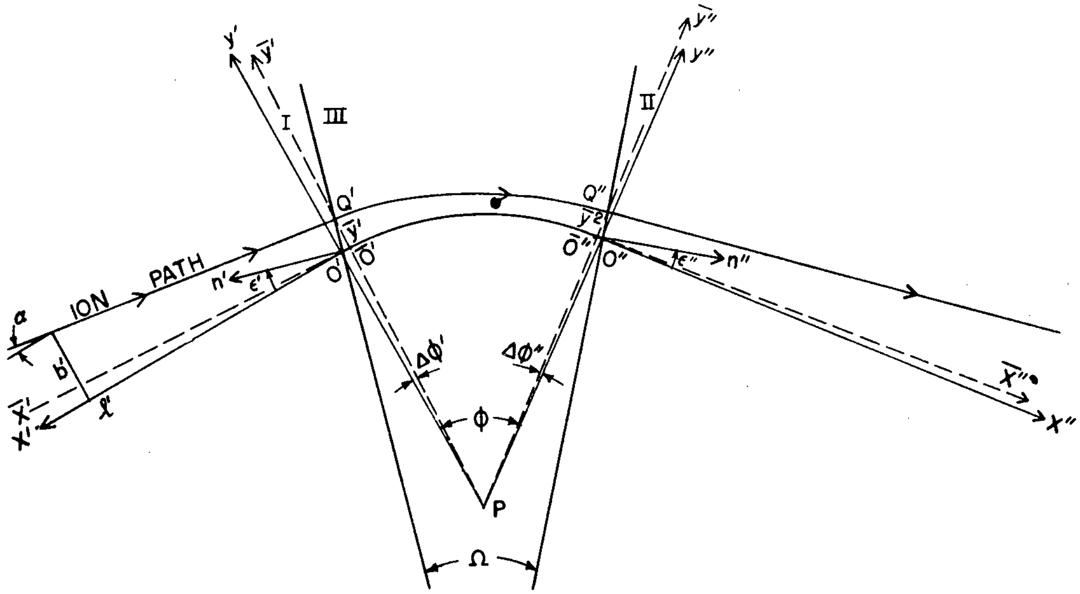
y′
ȳ′
I
III
II
ȳ″
y″
Q′
Q″
ȳ′
ȳ2
Ō′
O′
Ō″
O″
ION
PATH
n′
ε′
ε″
n″
α
b′
ℓ′
X̄′
X′
ΔΦ′
ΔΦ″
Φ
P
Ω
X̄″
X″

Fig. 2

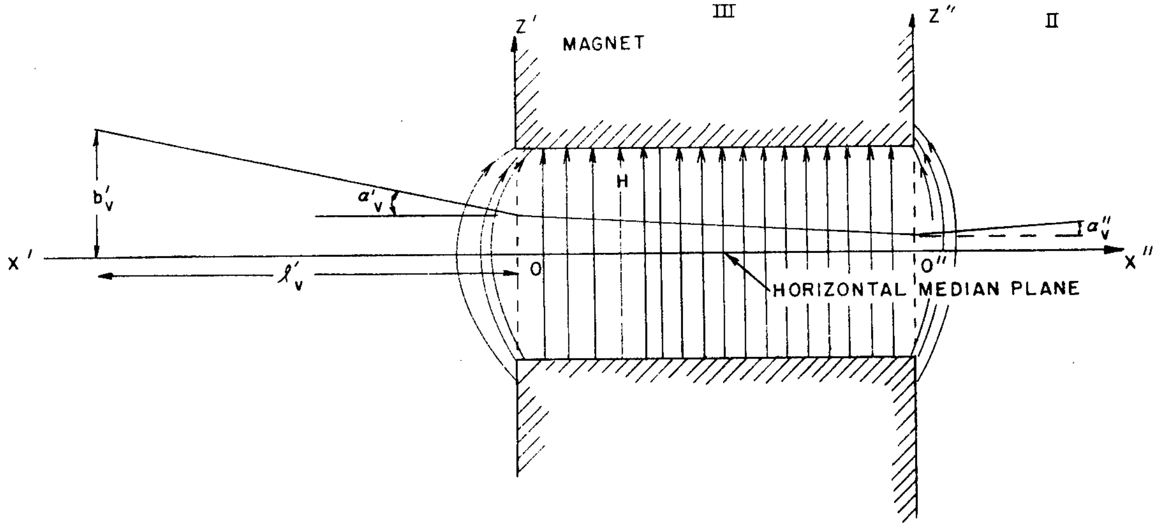
Z'
MAGNET
III
Z''
II
H
b'_v
a'_v
a''_v
X'
ℓ'_v
O
O''
X''
HORIZONTAL MEDIAN PLANE

Fig. 3

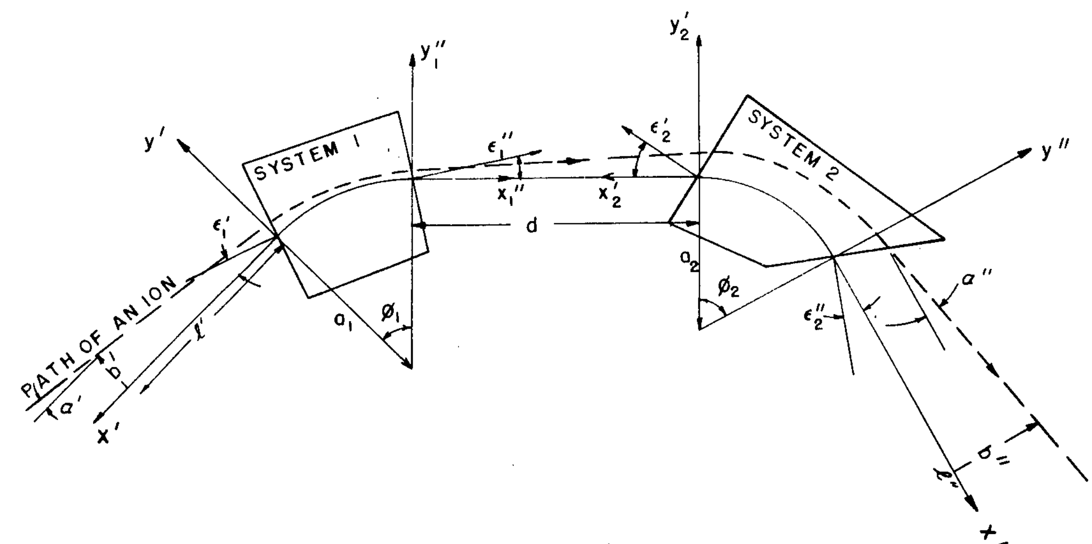
y''_1
y'_2
y'
SYSTEM 1
SYSTEM 2
y''
ϵ''_1
ϵ'_2
x''_1
x'_2
d
ϵ'_1
a_2
ϕ_2
a_1
ϕ_1
PATH OF AN ION
ℓ'
b'
a'
x'
ϵ''_2
a''
b''
ℓ''
x''

Fig. 4

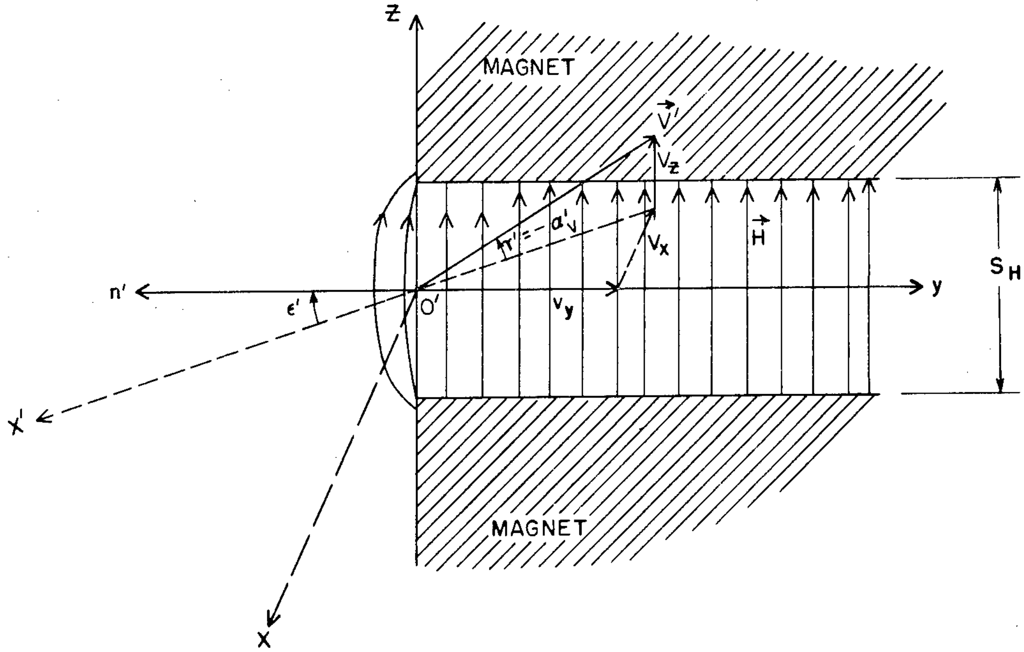

Fig. 5

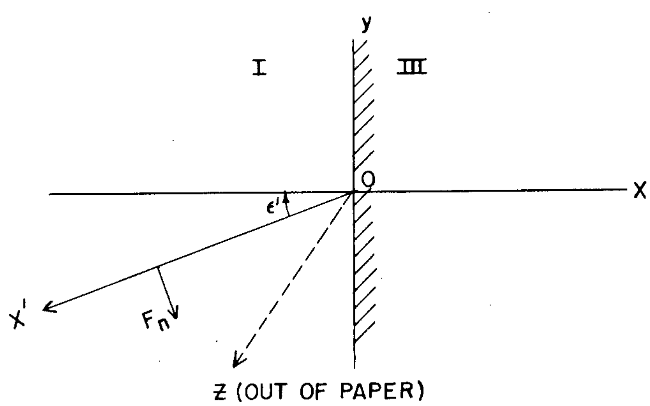

Fig. 6

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